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On conserved densities and asymptotic behaviour for the potential Kadomtsev–Petviashvili equation

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Abstract
We study local conservation laws with non-vanishing conserved densities and corresponding boundary conditions for the potential Kadomtsev–Petviashvili equation. We analyse an infinite symmetry group of the equation, and generate a finite number of conserved densities corresponding to infinite symmetries through appropriate boundary conditions.

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1. Introduction

The Kadomtsev–Petviashvili equation arises in many applications such as fluid dynamics (weakly long waves in shallow water, multi-layered shallow fluid), plasma physics, gas dynamics. Most known aspects of the study of the Kadomtsev–Pertviashvili (KP) equation are related to soliton solutions and the inverse scattering method (see, e.g., Ablowitz and Clarkson (1991)). The Lie point symmetry group for the KP equation was calculated in Schwartz (1982) and Tajiri *et al* (1982). Properties of the infinite-dimensional group of the KP equation, potential KP equation, and equations of KP hierarchy were discussed in a number of papers, e.g., David *et al* (1985, 1986, 1988) and Orlov and Winternitz (1997). Symmetries and constants of motion for the KP equation were investigated in Zakharov and Schulman (1980), Oevel and Fuchssteiner (1982), Infeld and Frycz (1983) (potential KP), Chen *et al* (1983, 1987) and Finkel and Fokas (2002), etc.

The goal of the present paper is out of the infinite set of continuity equations for the potential Kadomtsev–Pertviashvili equation to find those conservation laws that lead to non-vanishing conserved densities (essential conservation laws). Our essential conservation laws are associated with the infinite Lie symmetry group of the original equation. For each essential conservation law of the potential KP equation we will identify corresponding boundary conditions (asymptotic behaviour). In our derivation we will follow the approach developed in Rosenhaus (2002).

A relationship between variational symmetries and conservation laws has a long history and goes back to the classic Noether results (Noether 1918) (see also Olver (1986)). According to the second Noether theorem (Noether 1918), infinite variational symmetries with arbitrary functions of all independent variables do not lead to conservation laws but to a certain relation between equations of the original differential system. Infinite variational symmetries with arbitrary functions of not all independent variables were shown to lead to a finite number of essential local conservation laws (Rosenhaus 2002). For infinite symmetries containing arbitrary functions of t, it was shown in Rosenhaus (2002) that the main factor determining the existence of corresponding conservation laws is the form of boundary conditions (see also Rosenhaus (2003, 2005)).

2. Infinite symmetries and essential conservation laws

By a conservation law for a differential system

$$\omega^a(x, u, u_i, \ldots) = 0, \qquad i = 1, \ldots, m+1, \qquad a = 1, \ldots, n, \qquad u_i^a \equiv \partial u^a/\partial x^i$$

is meant a continuity equation

$$D_i K_i \doteq 0, \qquad K_i = K_i(x, u, u_j \ldots), \qquad i, j = 1, \ldots, m+1, \qquad x^i = (x^1, x^2, \ldots x^m, t)$$

(K_i are smooth functions) which is satisfied for any solutions of the original system (Olver 1986). Each conservation law is defined up to an equivalence transformation $K_i \to K_i + P_i$, where $D_i P_i \doteq 0$. Two conservation laws belong to the same equivalence class if they differ by a trivial conservation law. For trivial conservation laws the components of the vector K_i vanish on the solutions, $K_i \doteq 0$, ($i = 1, \ldots, m+1$), or the continuity equation is satisfied in the whole space, $D_i K_i = 0$ (Olver 1986). By an *essential* conservation law (Rosenhaus 2003), we mean such a non-trivial conservation law $D_i K_i \doteq 0$, which gives rise to a non-vanishing conserved density

$$D_t \int_D K_t \, \mathrm{d}x^1 \, \mathrm{d}x^2 \cdots \mathrm{d}x^m \doteq 0, \qquad x \in D \subset R^{m+1}, \qquad K_t \not\doteq 0. \tag{1}$$

We consider functions $u = u(x)$ defined on a region D of $(m+1)$-dimensional spacetime. Let

$$S = \int_D L\big(x^i, u^a, u_i^a, \ldots\big) \, \mathrm{d}^{m+1}x \qquad a = 1, \ldots, n, \qquad i, j = 1, \ldots, m+1$$

be the action functional, where L is the Lagrangian density. The equations of motion are

$$E^a(L) \equiv \omega^a(x, u, u_i, u_{ij}, \ldots) = 0, \qquad a = 1, \ldots, n, \qquad i, j = 1, \ldots, m+1, \tag{2}$$

where E is the Euler–Lagrange operator

$$E^a = \frac{\partial}{\partial u^a} - \sum_i D_i \frac{\partial}{\partial u_i^a} + \sum_{i \leqslant j} D_i D_j \frac{\partial}{\partial u_{ij}^a} + \cdots. \tag{3}$$

Consider an infinitesimal transformation with the canonical operator

$$\begin{aligned} X_\alpha &= \alpha^a \frac{\partial}{\partial u^a} + (D_i \alpha^a) \frac{\partial}{\partial u_i^a} + \sum_{i \leqslant j} (D_i D_j \alpha^a) \frac{\partial}{\partial u_{ij}^a} + \cdots, \\ \alpha^a &= \alpha^a(x, u, u_i, \ldots) \qquad i, j = 1, \ldots, m+1, \qquad a = 1, \ldots, n \end{aligned} \tag{4}$$

(summation over repeated indices is assumed). Variation of the functional S under the transformation with operator X_α is

$$\delta S = \int_D X_\alpha L \, \mathrm{d}^{m+1}x. \tag{5}$$

X_α is a variational (Noether) symmetry if

$$X_\alpha L = D_i M_i, \qquad M_i = M_i(x, u, u_i, \ldots), \qquad i = 1, \ldots, m+1, \tag{6}$$

where M_i are smooth functions. In the future we will use the Noether identity (Rosen 1972) (see also, e.g., Ibragimov (1985) or Rosenhaus (2002):

$$X_\alpha = \alpha^a E^a + D_i R_{\alpha i}, \qquad i = 1, \ldots, m+1, \qquad a = 1, \ldots, n, \tag{7}$$

$$R_{\alpha i} = \alpha^a \frac{\partial}{\partial u_i^a} + \left\{ \sum_{k \geqslant i} (D_k \alpha^a) - \alpha^a \sum_{k \leqslant i} D_k \right\} \frac{\partial}{\partial u_{ik}^a} + \cdots. \tag{8}$$

Applying the Noether identity (7) (with (8)) to L, and combining with (6), we obtain

$$D_i(M_i - R_{\alpha i} L) = \alpha^a \omega^a, \qquad i = 1, \ldots, m+1, \qquad a = 1, \ldots, n. \tag{9}$$

Equation (9) applied on the solution manifold ($\omega = 0, D_i \omega = 0, \ldots$) leads to a continuity equation

$$D_i(M_i - R_{\alpha i} L) \doteq 0, \qquad i = 1, \ldots, m+1. \tag{10}$$

Thus, any 1-parameter variational symmetry transformation α (6) leads to a conservation law (10) (the first Noether theorem) with the characteristic α. The second Noether theorem (Noether 1918) deals with a case of an infinite variational symmetry group where the symmetry vector α is of the form

$$\alpha = ap(x) + b_i D_i p(x) + c_{ij} D_i D_j p(x) + \cdots, \tag{11}$$

and $p(x)$ is an arbitrary function of all base variables of the space. Unlike with the first Noether theorem, a consequence of an infinite symmetry (11) of functional S is not a conservation law but a certain relation between the original differential equations (Noether 1918). A general situation when $p(x)$ is an arbitrary function of not all base variables was analysed in Rosenhaus (2002). For a Noether symmetry transformation X_α we have

$$\delta S = \int_D \delta L \, \mathrm{d}^{m+1}x = \int_D X_\alpha L \, \mathrm{d}^{m+1}x = \int_D D_i M_i \, \mathrm{d}^{m+1}x = 0, \qquad x \in D \subset R^{m+1}. \tag{12}$$

Therefore, the following conditions for M_i (Noether boundary conditions) should hold (Rosenhaus 2002):

$$M_i(x, u, \ldots)|_{x^i \to \partial D} = 0, \qquad \forall i = 1, \ldots, m+1. \tag{13}$$

Equations (13) are usually satisfied for a 'regular' asymptotic behaviour $u, u_i \to 0$ as $x \to \pm\infty$, or for periodic solutions. Let us consider now another type of boundary conditions related to the existence of local conserved quantities. Integrating equation (10) over the space $(x^1, x^2, \ldots, x^m)$ we get

$$\int \mathrm{d}x^1 \, \mathrm{d}x^2 \cdots \mathrm{d}x^m \, D_t(M_t - R_{\alpha t} L) \doteq \int \mathrm{d}x^1 \cdots \mathrm{d}x^m \sum_{i=1}^{m} D_i(R_{\alpha i} L - M_i). \tag{14}$$

Applying the Noether boundary condition (13) and requiring the LHS of (14) to vanish on the solution manifold we obtain the 'strict' boundary conditions (Rosenhaus 2002)

$$R_{\alpha 1} L|_{x^1 \to \partial D} = R_{\alpha 2} L|_{x^2 \to \partial D} = \cdots = R_{\alpha m} L|_{x^m \to \partial D} = 0. \tag{15}$$

In this paper, we will be mainly interested in symmetries with arbitrary functions of time $\gamma(t)$. It is easy to demonstrate that infinite symmetries with arbitrary functions of t can lead only to a finite number of essential conservation laws for equations with first-order Lagrangian functions, $L = L(u, u_x, u_t)$; for details and a generalization to higher order Lagrangians, see Rosenhaus (2002). Consider variational symmetry α of the form

$$\alpha = a\gamma(t) + b\gamma'(t) + c\gamma''(t) + \cdots + h\gamma^{(l)}(t). \tag{16}$$

In order for a differential system to possess Noether local conserved quantities, both Noether (13) and strict boundary conditions (15) have to be satisfied. The corresponding Noether conservation law can be found in the form

$$D_t \int \mathrm{d}x^1\, \mathrm{d}x^2 \cdots \mathrm{d}x^m (M_t - R_{\alpha t} L) \doteq 0. \tag{17}$$

Writing M_t as

$$M_t = A\gamma(t) + B\gamma'(t) + C\gamma''(t) + \cdots + H\gamma^{(l)}(t), \tag{18}$$

from (17) we obtain

$$D_t \int \mathrm{d}x^1\, \mathrm{d}x^2 \cdots \mathrm{d}x^m [\gamma(t)A_1 + \gamma'(t)A_2 + \cdots + \gamma^{(l)}(t)A_l] \doteq 0, \tag{19}$$

where

$$A_1 = \left(A - a\frac{\partial L}{\partial u_t}\right), \qquad A_2 = \left(B - b\frac{\partial L}{\partial u_t}\right), \ldots, \qquad A_l = \left(H - h\frac{\partial L}{\partial u_t}\right).$$

Since $\gamma(t)$ is arbitrary we get

$$\int \mathrm{d}x^1\, \mathrm{d}x^2 \cdots \mathrm{d}x^m \left(A - a\frac{\partial L}{\partial u_t}\right) \doteq \int \mathrm{d}x^1\, \mathrm{d}x^2 \cdots \mathrm{d}x^m \left(B - b\frac{\partial L}{\partial u_t}\right) \doteq \cdots$$
$$\doteq \int \mathrm{d}x^1\, \mathrm{d}x^2 \cdots \mathrm{d}x^m \left(H - h\frac{\partial L}{\partial u_t}\right) \doteq 0. \tag{20}$$

Obviously, equations (20), in general, do not determine a system of conservation laws but impose additional constraints. Thus, Noether symmetries with arbitrary functions of time instead of conservation laws lead to a set of additional constraints imposed on the function u and its derivatives. Therefore, the satisfaction of the strict boundary conditions (15), along with the Noether boundary conditions (13), becomes critical in the sense of avoiding additional constraints (20). Correspondingly, we have three possible situations:

(1) The strict boundary conditions (15) along with the Noether boundary conditions (13) can be satisfied for arbitrary function $\gamma(t)$. Then the system (20) instead of conservation laws provides additional constraints that the function u and its derivatives must satisfy.

(2) The strict boundary conditions (15) along with the Noether boundary conditions (13) can be satisfied for some particular functions $\gamma(t)$. In this case the (finite) symmetry (16) leads to the Noether conservation law (17) in agreement with the first Noether theorem.

(3) The strict boundary conditions (15) cannot be satisfied for any functions $\gamma(t)$. In this case a consequence of an infinite symmetry (16) will be the fact that the solutions of the original differential equation with the boundary conditions (13) and (15) do not exist.

Thus, in order to avoid additional constraints (20) we have to find those particular functions $\gamma(t)$ that lead to different boundary conditions than the ones in the general case when the function $\gamma(t)$ is arbitrary (Rosenhaus 2002). Each choice of such functions $\gamma(t)$ gives rise to a respective conserved quantity.

3. Essential conservation laws for the potential KP equation

Let us apply the approach above for finding non-vanishing conserved densities of the potential Kadomtsev–Petviashvili equation (21) with boundary conditions on the infinity

$$u_{xxxx} + 6u_x u_{xx} + 3s^2 u_{yy} + 4u_{xt} = 0, \qquad s^2 = \mp 1 \tag{21}$$

(KP-I and KP-II, correspondingly). The Lagrangian function of equation (21) is

$$L = \frac{u_{xx}^2}{2} - u_x^3 - 2u_x u_t - \frac{3s^2 u_y^2}{2}. \tag{22}$$

The following operators determine the Lie point symmetry group of equation (21) (David *et al* 1986):

$$\begin{aligned}
X_\gamma &= \gamma \frac{\partial}{\partial x} + [2x\gamma'/3 - 4s^2 y^2 \gamma''/9] \frac{\partial}{\partial u}, \\
X_g &= g \frac{\partial}{\partial y} - (2s^2 y g'/3) \frac{\partial}{\partial x} + [-4s^2 x y g''/9 + 8y^3 f'''/81] \frac{\partial}{\partial u}, \\
X_f &= f \frac{\partial}{\partial t} + (2yf'/3) \frac{\partial}{\partial y} + [xf'/3 - 2s^2 y^2 f''/9] \frac{\partial}{\partial x} \\
&\qquad + [-uf'/3 + x^2 f''/9 - 4s^2 x y^2 f'''/27 + 4y^4 f^{(\mathrm{IV})}/243] \frac{\partial}{\partial u}, \\
X_h &= yh \frac{\partial}{\partial u}, \qquad X_l = l \frac{\partial}{\partial u},
\end{aligned} \tag{23}$$

where $\gamma(t), g(t), f(t), h(t), l(t)$ are arbitrary functions. Let us analyse infinite subalgebras of the symmetry algebra (23). First, let us write our symmetry operators in the canonical form. For an operator

$$X = \xi^t \frac{\partial}{\partial t} + \xi^x \frac{\partial}{\partial x} + \xi^y \frac{\partial}{\partial y} + \cdots + \eta \frac{\partial}{\partial u}, \tag{24}$$

a corresponding canonical operator takes a form

$$X_\alpha = X - \xi^i D_i = \alpha \frac{\partial}{\partial u} + \zeta_i \frac{\partial}{\partial u_i} + \sigma_{ij} \frac{\partial}{\partial u_{ij}} \cdots, \tag{25}$$

where

$$\alpha = \eta - \xi^i u_i, \qquad \zeta_i = D_i \alpha, \qquad \sigma_{ij} = D_{ij}\alpha, \qquad \ldots. \tag{26}$$

We will start with the symmetry operator X_γ and find corresponding conserved densities (Rosenhaus 2003, 2005).

3.1. Conserved densities associated with X_γ

Using (26), (24) and (16) we get

$$\begin{aligned}
&\xi^x = \gamma, \qquad \xi^t = \xi^y = 0, \qquad \eta = 2x\gamma'/3 - 4s^2 y^2 \gamma''/9, \\
&\alpha = -\gamma u_x + 2x\gamma'/3 - 4s^2 y^2 \gamma''/9, \qquad a = -u_x, \\
&b = 2x/3, \qquad c = -4s^2 y^2/9.
\end{aligned} \tag{27}$$

Calculating $X_\alpha L$ we obtain

$$X_\alpha L = D_x(-\gamma L - 4xu\gamma''/3 + 8s^2 y^2 u\gamma'''/9) + D_y(8s^2 yu\gamma''/3) - D_t(4u\gamma'/3). \tag{28}$$

Thus, X_γ is a Noether symmetry operator and using (6) and (18) we get

$$
\begin{aligned}
&X_\alpha L = D_i M_i, \qquad M_t = -4u\gamma'/3,\\
&M_x = -\gamma L - 4xu\gamma''/3 + 8s^2y^2u\gamma'''/9, \qquad M_y = 8s^2yu\gamma''/3,\\
&A = C = 0, \qquad B = -4u/3.
\end{aligned} \tag{29}
$$

The form of Noether and strict boundary conditions depends on the function $\gamma(t)$.

(A) $\gamma(t)$ *is arbitrary*

The Noether boundary conditions (13) for X_α are

$$xu, (L) \underset{x\to\pm\infty}{\rightarrow} 0, \qquad yu \underset{y\to\pm\infty}{\rightarrow} 0, \qquad \gamma' u \underset{t\to\pm\infty}{\rightarrow} 0. \tag{30}$$

The strict boundary conditions (15) take the form

$$xu_t, xu_x^2, xu_{xxx}, u_{xx} \underset{x\to\pm\infty}{\rightarrow} 0, \qquad y^2u_y, u_xu_y \underset{y\to\pm\infty}{\rightarrow} 0. \tag{31}$$

The symmetry transformation X_γ for arbitrary $\gamma(t)$ leads to a system of additional constraints (20) instead of conservation laws.

In order to avoid restrictions (20) let us consider some specific forms of $\gamma(t)$ for which we can weaken our boundary conditions (30)–(31) and (20).

(B) $\gamma'(t) = 0,\ \gamma(t) = \text{const}$.

In this case α and M_i simplify to

$$\alpha = -u_x\gamma, \qquad M_x = -\gamma L, \qquad M_y = M_t = 0. \tag{32}$$

Noether boundary conditions are

$$u_x, u_y, u_xu_t, u_{xx} \underset{x\to\pm\infty}{\rightarrow} 0. \tag{33}$$

Strict boundary conditions, in addition to (33), will have

$$u_{xxx} \underset{x\to\pm\infty}{\rightarrow} 0, \qquad u_xu_y \underset{y\to\pm\infty}{\rightarrow} 0. \tag{34}$$

According to (17), a symmetry X_γ in this case will lead to the following essential conservation law:

$$D_t \iint u_x^2 \,\mathrm{d}x\,\mathrm{d}y \doteq 0, \tag{35}$$

or

$$\iint u_x^2 \,\mathrm{d}x\,\mathrm{d}y \doteq \text{const}$$

(conservation of the x-component of the momentum P_x) with the boundary conditions (32)–(33). The corresponding continuity equation has the form

$$D_x\left(2u_x^3 + u_xu_{xxx} - u_{xx}^2/2 - 3u_y^2/2\right) + D_y(3u_xu_y) + D_t\left(2u_x^2\right) \doteq 0. \tag{36}$$

(C) $\gamma''(t) = 0,\ \gamma'(t) \neq 0$: $\gamma(t) = at,\ a = \text{const} \neq 0$

We have

$$\alpha = -atu_x + 2ax/3, \qquad M_x = -atL, \qquad M_y = 0, \qquad M_t = -4au/3. \tag{37}$$

The Noether boundary conditions are the same as in case B (33) plus

$$u \underset{t\to\pm\infty}{\rightarrow} 0. \tag{38}$$

For strict boundary conditions in addition to (34) we get

$$xu_t,\, xu_x^2,\, xu_{xxx} \underset{x\to\pm\infty}{\to} 0, \qquad u_y \underset{y\to\pm\infty}{\to} 0. \tag{39}$$

The essential conservation law associated with the boundary conditions (38), (39), (33), (34) takes the form

$$D_t \iint \left[2tu_x^2 - 4(xu_x - u)/3\right] \mathrm{d}x\,\mathrm{d}y \doteq 0. \tag{40}$$

(D) $\gamma''(t) \neq 0$

In this case Noether and strict boundary conditions have the same form (30), (31) as in case A and lead to no essential conservation laws.

3.2. Conserved densities associated with X_g

For a corresponding canonical operator X_α we get

$$\begin{aligned} &\alpha = -gu_y + 2s^2yu_xg'/3 - 4s^2xyg''/9 + 8y^3g'''/81, \\ &\xi^x = -2s^2yg'/3, \quad \xi^y = g, \quad \eta = -4s^2xyg''/9 + 8y^3g'''/81. \end{aligned} \tag{41}$$

Calculating $X_\alpha L$ we obtain

$$\begin{aligned} &X_\alpha L = D_i M_i, \\ &M_x = 2s^2yLg'/3 + 8s^2x^2ug'''/9 - 16y^3ug^{(\mathrm{IV})}/81, \\ &M_y = -gL + 4xug''/3 - 8s^2y^2ug'''/9, \\ &M_t = 8s^2yug''/9. \end{aligned} \tag{42}$$

As in the previous case, the forms of strict and Noether boundary conditions depend on the function $g(t)$.

(A) $g(t)$ is arbitrary

From the Noether and strict boundary conditions we get

$$x^2u,\, (L) \underset{x\to\pm\infty}{\to} 0, \qquad y^2u,\, y^3u_y,\, (L) \underset{y\to\pm\infty}{\to} 0, \qquad g''u \underset{t\to\pm\infty}{\to} 0. \tag{43}$$

No local conservation laws are associated with the Noether transformation X_g when $g(t)$ is arbitrary. Let us consider now some specific forms of function $g(t)$ for which we can weaken boundary conditions (43).

(B) $g' = 0, \quad g(t) = c = \mathrm{const}.$

We have

$$\alpha = -cu_y, \qquad M_x = M_t = 0, \qquad M_y = -cL. \tag{44}$$

Noether boundary conditions look as follows:

$$u_x,\, u_y,\, u_xu_t,\, u_{xx} \underset{y\to\pm\infty}{\to} 0. \tag{45}$$

The strict boundary conditions have a form

$$u_x,\, u_y,\, u_t,\, u_{xx},\, u_{xy},\, u_{xxx} \underset{x\to\pm\infty}{\to} 0, \qquad u_y \underset{y\to\pm\infty}{\to} 0. \tag{46}$$

According to (17), the associated conservation law has a form

$$D_t \iint u_xu_y \,\mathrm{d}x\,\mathrm{d}y \doteq 0. \tag{47}$$

Expression (47) is a conservation of the y-component of the momentum of the system P_y with the regular boundary conditions (45)–(46).

(C) $g'' = 0,\ \ g' \neq 0: \quad g(t) = at,\ a = \text{const} \neq 0$

We have

$$\alpha = -atu_y + 2as^2yu_x/3, \qquad M_x = 2as^2yL/3, \qquad M_y = -atL, \qquad M_t = 0. \tag{48}$$

Noether boundary conditions are

$$u_x, u_y, u_xu_t, u_{xx} \underset{x\to\pm\infty}{\to} 0, \qquad u_x, u_y, u_xu_t, u_{xx} \underset{y\to\pm\infty}{\to} 0, \tag{49}$$

and strict boundary conditions in addition to (49) have

$$u_{xy}, u_{xxx} \underset{x\to\pm\infty}{\to} 0, \qquad yu_xu_y \underset{y\to\pm\infty}{\to} 0. \tag{50}$$

The boundary conditions (49)–(50) are weaker than the ones for the general case (43), and according to (17), we obtain the following conserved quantity:

$$D_t \iint \left[3tu_xu_y - 2s^2yu_x^2\right] \mathrm{d}x\, \mathrm{d}y \doteq 0. \tag{51}$$

(D) $g''' = 0,\ \ g'' \neq 0: \qquad g(t) = bt^2/2, \qquad b = \text{const} \neq 0$

We have

$$\alpha = -bt^2u_y/2 + 2s^2btyu_x/3 - 4s^2bxy/9, \qquad M_x = 2s^2btyL/3,$$
$$M_y = -bt^2L/2 + 4bxu/3, \qquad M_t = 8s^2byu/9. \tag{52}$$

Noether boundary conditions have in addition to (49)

$$u \underset{y\to\pm\infty}{\to} 0, \qquad u \underset{t\to\pm\infty}{\to} 0, \tag{53}$$

and strict boundary conditions in addition to (50) read

$$xu_t, xu_x^2, xu_{xxx} \underset{x\to\pm\infty}{\to} 0, \qquad yu_y \underset{y\to\pm\infty}{\to} 0. \tag{54}$$

The boundary conditions for this case—(53), (54), (49) and (50)—are weaker than in the general case (43). The following conservation law (17) is associated with the symmetry X_g in this case:

$$D_t \iint \left[t^2u_xu_y - 4ts^2yu_x^2/3 + 8s^2y(xu_x - u)/9\right] \mathrm{d}x\, \mathrm{d}y \doteq 0. \tag{55}$$

(D) $g^{(\mathrm{IV})} = 0,\ \ g''' \neq 0: g(t) = kt^3/3, k = \text{const} \neq 0$

We have

$$\alpha = -kt^3u_y/3 + 2s^2kt^2yu_x/3 - 8s^2kxyt/9 + 16ky^3/81, \qquad M_t = 16s^2kyu/9.$$
$$M_x = 2s^2kt^2yL/3 + 16s^2kx^2u/9, \qquad M_y = -kt^3L/3 + 8kxut/3 - 16s^2ky^2u/9. \tag{56}$$

Noether boundary conditions have in addition to (53) and (49)

$$x^2u \underset{x\to\pm\infty}{\to} 0, \qquad y^2u \underset{y\to\pm\infty}{\to} 0, \tag{57}$$

and strict boundary conditions in addition to (54) and (50) read

$$xu_t, xu_x^2, xu_{xxx} \underset{x\to\pm\infty}{\to} 0, \qquad y^3u_y \underset{y\to\pm\infty}{\to} 0. \tag{58}$$

The boundary conditions for this case, (57) and (58) (with (49), (50), (53) and (54)) are the same as in the general case (43). No essential conservation laws are associated with the symmetry X_g.

(E) $g^{\mathrm{IV}}(t) \neq 0$

The same boundary conditions as in case A. The symmetry X_g leads to no essential conservation laws.

3.3. Conserved densities associated with X_f

For X_α in this case we obtain

$$\alpha = -fu_t - f'[2yu_y + xu_x + u]/3 + f''(x^2 + 2s^2y^2u_x)/9 - 4s^2xy^2f'''/27 + 4y^4f^{(\mathrm{IV})}/243. \tag{59}$$

Calculating $X_\alpha L$ we obtain

$$\begin{aligned} X_\alpha L &= D_i M_i, \\ M_x &= -xLf'/3 + [2s^2y^2L/9 + u^2/3 - 2u_x/9]f'' - 2ux^2f'''/9 \\ &\quad + 8s^2xy^2uf^{(\mathrm{IV})}/27 - 16y^4uf^{(\mathrm{V})}/243, \\ M_y &= -2yLf'/3 + 8xyuf'''/9 - 16s^2y^3uf^{(\mathrm{IV})}/81, \\ M_t &= -fL - 4xuf''/9 + 8s^2y^2uf'''/27. \end{aligned} \tag{60}$$

As in the previous case, the forms of strict and Noether boundary conditions depend on the function $f(t)$.

(A) $f(t)$ is arbitrary

From the Noether and strict boundary conditions we will get

$$x^2u \underset{x\to\pm\infty}{\to} 0, \qquad y^3u,\, y^4u_y \underset{y\to\pm\infty}{\to} 0, \qquad f''u,\, f'''u,\, fL \underset{t\to\pm\infty}{\to} 0. \tag{61}$$

No local conservation laws are associated with the Noether transformation X_α (59) with arbitrary function $f(t)$. Let us consider now some specific forms of $f(t)$ for which we can weaken the boundary conditions (61).

(B) $f' = 0, \quad f(t) = c = \text{const}.$

We have

$$\alpha = -cu_t, \qquad M_x = M_y = 0, \qquad M_t = -cL. \tag{62}$$

Noether and strict conditions look as follows:

$$u_x, u_t, u_{xx}, u_{xt}, u_{xxx} \underset{x\to\pm\infty}{\to} 0, \qquad u_tu_y \underset{y\to\pm\infty}{\to} 0, \qquad L \underset{t\to\pm\infty}{\to} 0. \tag{63}$$

The associated conservation law has a form

$$D_t \iint [u_{xx}^2 - 2{u_x}^3 - 3s^2{u_y}^2]\mathrm{d}x\,\mathrm{d}y \doteq 0. \tag{64}$$

Expression (64) is a conservation of energy of the system, corresponding to the regular boundary conditions (63).

(C) $f'' = 0,\, f' \neq 0 : f(t) = at,\, a = \text{const} \neq 0.$

We have

$$\begin{aligned} &\alpha = -a(3tu_t + 2yu_y + xu_x - u)/3, \qquad M_x = -axL/3, \\ &M_y = -2ayL/3, \qquad M_t = -atL. \end{aligned} \tag{65}$$

Noether and strict boundary conditions here in addition to (63) are

$$xL,\ xu_xu_{xxx},\ uu_x{}^2,\ uu_t,\ uu_{xxx} \underset{x\to\pm\infty}{\to} 0, \qquad yL,\ u_tu_y,\ uu_y \underset{y\to\pm\infty}{\to} 0, \qquad tL \underset{t\to\pm\infty}{\to} 0. \tag{66}$$

The boundary conditions (66) are weaker than the ones for the general case (61), and we obtain the following conserved quantity:

$$D_t \iint [3tL + 2u_x(u + xu_x + 2yu_y + 3tu_t)]\, \mathrm{d}x\, \mathrm{d}y \doteq 0. \tag{67}$$

(D) $f''' = 0,\ f'' \neq 0 : f(t) = bt^2/2,\ b = \text{const} \neq 0.$

Noether and strict boundary conditions in this case have in addition to (66)

$$u,\ xu_x,\ xu_{xx},\ x^2u_t,\ x^2u_{xxx} \underset{x\to\pm\infty}{\to} 0, \qquad y^2u_xu_y \underset{y\to\pm\infty}{\to} 0, \qquad t^2L \underset{t\to\pm\infty}{\to} 0. \tag{68}$$

The boundary conditions (68) are weaker than in the general case (61), and the symmetry X_f leads to the following conservation law:

$$D_t \iint \left[9t^2L + 6tu_x(2u + 2xu_x + 4yu_y + 3tu_t) + \left(8xu - 4x^2u_x - 8s^2y^2u_x^2\right)\right] \mathrm{d}x\, \mathrm{d}y \doteq 0. \tag{69}$$

(E) $f^{(\mathrm{IV})} = 0,\ f''' \neq 0 : f(t) = kt^2/2,\ k = \text{const} \neq 0$

Noether and strict boundary conditions in this case have in addition to (68)

$$x^2u \underset{x\to\pm\infty}{\to} 0, \qquad yu,\ y^2u_y \underset{y\to\pm\infty}{\to} 0, \qquad tu,\ t^3L \underset{t\to\pm\infty}{\to} 0. \tag{70}$$

The boundary conditions are still weaker than in the general case (61), and the symmetry X_f leads to the following conservation law:

$$D_t \iint \left[t^3L + 2t^2u_x(u + xu_x + 2yu_y + tu_t) + \frac{8}{3}t\left(2xu - s^2y^2u_x^2\right) - \frac{32}{9}s^2y^2u\right] \mathrm{d}x\, \mathrm{d}y \doteq 0. \tag{71}$$

(F) $f^{(\mathrm{V})} = 0, \quad f'''' \neq 0 : \quad f(t) = lt^4/4, \quad l = \text{const} \neq 0$

Noether and strict boundary conditions in this case have the same form (61) as for the general case, and the symmetry X_f leads to no essential conservation laws.

(G) $f^{(\mathrm{V})} \neq 0$

No essential conservation laws are associated with the symmetry X_f.

Infinite symmetries X_h and X_l do not lead to any essential conservation laws.

4. Conclusions

We have generated a finite set of essential conserved densities for the potential Kadomtsev–Petviashvili equation (21) associated with its infinite classical symmetry group.

In an elegant paper by Infeld and Frycz (1983) an infinite set of continuity equations for the potential KP equation was presented. In this paper we identified those continuity equations that lead to non-vanishing (Noether) conserved densities (essential conservation laws). For the potential Kadomtsev–Petviashvili equation it was demonstrated that infinite symmetries with arbitrary functions of t lead to a finite number of essential local conservation laws only in special cases when boundary conditions are weaker than those in the general case. Each conservation law is determined by a specific form of boundary condition, and known conservation laws of momentum and energy (35), (47), (64) correspond to the weakest boundary conditions. Other essential local conservation laws assume stricter asymptotic behaviour.

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